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A LINEAR ANALYSIS APPROACH TO THE SOLUTION OF CERTAIN CLASSES OF VARIATIONAL INEQUALITY PROBLEMS IN STRUCTURAL ANALYSIS

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Abstract—In this paper a method is presented, which provides a reliable tool for the solution of unilateral problems arising in structural mechanics. The method is based on some theorems of quadratic programing and combines the advantages of the optimization algorithms (systematic choice of the iteration steps and convergence) with the advantages of "trial and error" methods (use of general purpose programs which treat effectively large problems). The method is extended to cover also problems described by positive-semidefinite matrices. Test cases of frames, plates and shells demonstrate the applicability and the convergence of the method.

1. INTRODUCTION

Structures frequently contain members capable of transmitting only certain types of stress, e.g. only tension. In other structures the boundary tractions or geometrical constraints are imposed in the form of inequalities, e.g. a tensionless support. Finally, the yield criteria in the theory of plasticity introduce inequalities in the formulation of the problem.

Such so-called unilateral problems are inherently nonlinear, even in cases of linear elasticity and small deformations. In general, the unilateral problems are characterized by the fact that changes of the external loadings are not accompanied by proportional changes in other variables. Consequently, superposition is not, generally, applicable.

Existing analytical solutions are limited to idealized models and simple inequality conditions. In the case of contact problems the finite element method has been used for the numerical treatment of some practical problems. Many papers dealing with the static analysis of contact problems assume a contact zone or treat the problem by using iterative methods, the so called "trial and error" methods (see, e.g. [1–7]). The assumption of bilateral conditions on the contact zone restrict the applicability of the methods, because the contact area is not known *a priori*. Convergence to the correct solution by iterative procedures is not always guaranteed. In general, "solutions" which satisfy the constraints of a unilateral problem do not need be the desired (correct) solutions. This is demonstrated by examples in the present paper. Finally, the incremental approach to the unilateral problems is accompanied by superfluous computational effort and may lead to wrong solutions if the increments do not follow the structural changes. Therefore, a systematic method is needed, for complex structures with arbitrary conditions. The application of the finite element method to unilateral problems leads to a mathematical programing problem and requires the use of optimization techniques [8–15]. However the existing numerous optimization algorithms are subjected to a variety of limitations; most of them are restricted to relatively small problems. Moreover, the existing general purpose programs cannot be used without modifications. In this context it is mentioned that the penalty method, as an existing general purpose optimization method, can be applied to a series of unilateral problems. But, due to its generality, the required effort could be more than the effort required by special quadratic optimization algorithms suitable for problems arising in structural analysis. Many versions of the penalty method transform the nonlinear programing problem

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into a sequence of unconstrained problems by adding one or more functions of the constraints to the objective function and deleting the constraints as such. However, convergence of the penalty method (see, e.g.[23]) depends strongly on the choice of the coefficient of the penalty term (Lagrange multipliers).

In this paper a method is presented which provides a reliable tool for the solution of unilateral problems without using optimization algorithms. The method is based on some theorems proved for quadratic optimization problems by Theil and van de Panne[16]. By the interpretation and the application of these theorems the unilateral problem is "replaced" equivalently by a number of classical or bilateral problems. The theorems of Theil and van de Panne permit the control of the iterations in such a manner, that convergence to the solution of the unilateral problem is assured. The criteria developed by interpreting the aforementioned theorems are extended to cover also problems described by positive-semidefinite matrices. Moreover the resulting bilateral problems can be handled numerically by means of mixed finite element models, as well. The mixed element method offers promising alternatives in the treatment of plates and shells. Accordingly the present method could be seen as an attempt towards the use of mixed finite elements for the numerical solution of unilateral problems arising in structural mechanics.

The method can be used for the solution of all quadratic optimization problems arising in structural mechanics and in some sense combines the advantages of the optimization algorithms with the advantages of the linear analysis computer programs, i.e. effective treatment of large problems.

Test cases of frames, plates and shells demonstrate the applicability and the convergence of the method. Also, the results are compared with available analytical and numerical solutions. The mathematical derivations and proofs have been given previously [16, 17] and will not be repeated here. Only the essential features, physical interpretations and methods of implementation are presented.

2. METHOD OF SOLUTION

2.1 *Introductory remarks*

The local formulation of the problems mentioned in the previous chapter requires, in addition to the field equations of three-dimensional elasticity (equilibrium and kinematic equations, elasticity law, boundary conditions), also constraints in the form of inequalities describing the unilateral phenomenon. In the transformation of this formulation to a global form, i.e. a variational principle, unilateral variations of the field variables arise. The conditions expressing from the physical point of view the principal of virtual work hold now in an inequality form (variational inequality) and, for these new type of boundary-value problems, special considerations concerning the extremum properties of the potential and complementary energy are necessary[13]. The numerical treatment of these variational inequalities leads to the solution of the following minimization problem:

$$\min\left[Q(\mathbf{q})=\frac{1}{2}\mathbf{q}^T\mathbf{F}\mathbf{q}+\mathbf{q}^T\mathbf{p}\,\middle|\,\mathbf{a}_j^T\mathbf{q}\leq b_j;\, j\in S_0=\{1,2,\ldots,m\}\right], \tag{1}$$

where $\mathbf{F}$ denotes, for the present, a positive-definite matrix, e.g. a stiffness matrix, and the vectors $\mathbf{q}$ and $\mathbf{p}$ represent the unknown variables and the applied loads, respectively. Relations (1) describe a quadratic programing problem which can be solved by appropriate algorithms.

The following method is based on optimization theorems but also incorporates some features of "trial and error" methods. However, it clearly differs from usual "trial and error" procedures, because it provides for the choice of the iteration steps and contains a criterion to test the optimality.

2.2 *Definitions*

The following definitions and notations are introduced:

U.C., B.C. Each inequality of problem (1) is termed a Unilateral Constraint written U.C. This constraint is called Equality Constraint, if only the corresponding equality is satisfied. If a U.C. is removed, i.e. it holds in the form $\mathbf{a}_j^T\mathbf{q} \lesseqgtr b_j$, then we say that the U.C. is replaced by a Bilateral Constraint (B.C.).

$\hat{\mathbf{q}}$ Solution of problem (1).

$\hat{S}$ The set of indices of those restrictions, which $\hat{\mathbf{q}}$ satisfies in equality form, i.e. $\mathbf{a}_j^T\hat{\mathbf{q}} = b_j$ for every $j \in \hat{S} \subset S_0$.

$\dot{\mathbf{q}}$ Unrestricted minimum of problem (1), i.e. all U.C.'s are replaced by B.C.'s.

S A subset of S_0 ($S \subset S_0$).

$\mathbf{q}^S$ Solution of

$$\min\,[Q(\mathbf{q})|\mathbf{a}_j^T\mathbf{q} = b_j;\, j \in S \subset S_0]. \tag{2}$$

Problem (2) is governed by the system resulting after S of the U.C.'s are satisfied in equality form and $S_0 - S$ of the U.C.'s are replaced by B.C.'s.

$\{h\}$ A set of one index h.

$\emptyset$ The empty set.

$V(\mathbf{q})$ Set of indices i for which $\mathbf{a}_i^T\mathbf{q} > b_i$, i.e. $V(\mathbf{q})$ is the set of the restrictions which are violated by a solution $\mathbf{q}$.

$\bar{V}(\mathbf{q})$ Set of indices identifying those restrictions which a vector $\mathbf{q}$ either violates in the inequality form or satisfies in the corresponding equality form.

2.3 *Theorems and rules*

The following theorems are used as basis of the method. The proofs are given in [16, 17].

Theorem 1: Let $\hat{S}$ denoting the set of indices indicating those restrictions, which the solution $\hat{\mathbf{q}}$ of problem (1) satisfies in equality form, i.e.

$$\mathbf{a}_j^T\hat{\mathbf{q}} = b_j \quad \text{for} \quad j \in \hat{S}$$

and

$$\mathbf{a}_j^T\hat{\mathbf{q}} < b_j \quad \text{for} \quad j \notin \hat{S}.$$

Then it holds

$$\hat{\mathbf{q}} = \mathbf{q}^{\hat{S}}.$$

Theorem 2: If $\hat{\mathbf{q}} = \mathbf{q}^{\hat{S}}$ and $\hat{S} \neq \emptyset$, then it holds for all subsets $S \subset \hat{S}$, including $S = \emptyset$, that

$$h \in V(q^S)$$

for at least one $h \in (\hat{S} - S)$.

Theorem 3: If $V(\mathbf{q}^S) = \emptyset$ and $h \in V(\mathbf{q}^{S-\{h\}})$ for all $h \in S$, then $\mathbf{q}^S = \hat{\mathbf{q}}$.
In the above theorem $\mathbf{q}^{S-\{h\}}$ denotes the solution of the problem

$$\min\,[Q(\mathbf{q})|\mathbf{a}_j^T\mathbf{q} = b_j; \quad j \in S - \{h\} \subset S_0]. \tag{3}$$

For our purposes these theorems are interpreted and incorporated in three rules, which are applied to obtain successive solutions to conventional (bilateral) problems and, finally, the solution of the original (unilateral) problem:

Rule 1: If the solution of the unrestricted problem $\dot{\mathbf{q}}$ violates some of the restrictions, the $\hat{\mathbf{q}}$, i.e. the correct solution, must satisfy at least one of these restrictions in equality form.

Rule 2: If a solution $\mathbf{q}^S$ of a subsequent system (problem 2), violates some restrictions then, at least one of those restrictions, must also be enforced in equality form to obtain the optimum of the initial problem, i.e. the solution $\hat{\mathbf{q}}$.

Rule 3: A solution $\mathbf{q}^S$ which satisfies all restrictions of the original problem, coincides with the correct solution $\hat{\mathbf{q}}$, if, and only if, for every $h \in S$ the subsequent solutions $\mathbf{q}^{S-\{h\}}$ violate the restriction h omitted in the solution $\mathbf{q}^S$.

Rule 1 indicates the first step needed to modify the initial trial, the solution of the unrestricted problem. Rule 2 describes the way to proceed through successive steps. Rule 3 provides the test for the optimality of solutions $\mathbf{q}^S$.

2.4 *Solution procedure*

The method proposed here consists in the transformation of the problem constrained by inequalities into a number of problems governed by equalities. As a starting point, the solution $\mathring{\mathbf{q}}$ of the system resulting from the replacement of all U.C.'s by B.C.'s is obtained and subsequently the set $V(\mathring{\mathbf{q}})$. If $V(\mathring{\mathbf{q}}) = \emptyset$, the $\mathring{\mathbf{q}}$ represents the correct solution; but usually some of the constraints are violated. If $V(\mathring{\mathbf{q}}) \neq \emptyset$, then, by the stated Rule 1, the correct solution $\hat{\mathbf{q}}$ satisfies, in equality form, at least one of the restrictions violated. Taking the system used in the previous step, as a basis, we fulfill the restriction in equality form and again solve the modified system. If the solution $\mathbf{q}^S$ does not satisfy all restrictions, i.e. if $V(\mathbf{q}^S) \neq \emptyset$, then according to Rule 2, at least one restriction, in equality form, is added in order to approach the correct number of equalities ($\hat{S}$). The procedure is continued until a $\mathbf{q}^S$ is obtained with $V(\mathbf{q}^S) = \phi$. This solution is not necessarily the optimal one. The optimality can be tested by means of Rule 3. To this end, the equality constraints ($\{h\}$) are successively replaced by B.C.'s and solutions $\mathbf{q}^{S-\{h\}}$ are examined for violation of the restriction h. If, and only if, each restriction is violated for all $h \in S$, then the optimal solution is found. In cases of numerous constraints which occur in practice, i.e. in problems of an extended contact zone, several U.C.'s may be replaced by equality constraints simultaneously. This leads to a considerable reduction in computational effort. Also, the execution of the control according to Rule 3 is not necessarily accompanied by additional computational effort since the solutions $q^{S-\{h\}}$ are usually known from previous steps. Applications of this method are illustrated in the subsequent examples.

The steps described above in connection with Rules 1–3 reveal the advantages of the method:

1. The iterations are not arbitrary but are indicated by the prescribed criteria.

2. There exists a criterion, which indicates whether a particular solution is the optimal one. It should be pointed out that a vector $\mathbf{q}^S$ which does not violate some restrictions, i.e. $V(\mathbf{q}^S) = \emptyset$, need not be the correct solution. The importance of this is shown in the numerical examples presented here. This criterion distinguishes the proposed method from the usual "trial and error" methods and proves especially useful for problems with complicated contact zones.

3. The transformation of the general quadratic optimization problem to a succession of unconstrained, or equality constrained, minimization problems (which can be solved by the classical methods of linear analysis) allows the use of general purpose computer programs. They can be slightly modified in order to avoid superfluous effort during the iteration. Thus the solution of problems involving many unknowns and constraints is possible, whereas the efficient application of optimization algorithms is limited by the large number of unknowns. Moreover, the optimization techniques require usually modifications in existing programs and may be affected by inaccuracies.

4. The method can also be applied to the solution of problems expressed in terms of positive-semidefinite matrices.

5. No assumptions concerning the active inequalities (e.g. the contact zone) need be made.

6. Convergence is guaranteed since the method is based on theorems already proved for the quadratic optimization problem by Theil and van de Panne[17].

2.5 *Extension to positive-semidefinite matrices*

In case, that the minimization problem (1) contains a positive-semidefinite matrix $\mathbf{F}$, the foregoing method applies with the following modification of Theorem 3 and the corresponding test of optimality.

Theorem 3*a*: Let the matrix **F** of problem (1) be positive-semidefinite. For a $\mathbf{q}^S$ with $V(\mathbf{q}^S) = \emptyset$,

$$\mathbf{q}^S = \hat{\mathbf{q}},$$

if, and only if, $h \in \bar{V}(\mathbf{q}^{S-\{h\}})$ for every $h \in S$.

The theorem states, that the B.C.'s, signified by index h, either must not belong to the admissible domain of problem (1) or must lie on the boundary of the admissible domain, i.e. $\mathbf{a}_h^T\mathbf{q} \geq b_h$. This must be true for every equality constraint of the "S" structure.

Proof: Let $V(\mathbf{q}^S) = \emptyset$, i.e.

$$\{\mathbf{a}_j^T\mathbf{q} = b_j, j \in S; \mathbf{a}_j^T\mathbf{q} \leq b_j, j \in S_0 - S\}. \tag{4}$$

Since $\mathbf{q}^S$ is the solution of problem (2), it satisfies the stationarity conditions:

$$\left\{-(\mathbf{p} + \mathbf{F}\mathbf{q}^S) = \sum_{j \in S} \lambda_j^S \mathbf{a}_j; \mathbf{a}_j^T\mathbf{q}^S = b_j, j \in S\right\}, \tag{5}$$

where λ_j^S denote the Lagrangean multipliers. Further $\mathbf{q}^{S-\{h\}}$ satisfies the conditions:

$$\left\{-(\mathbf{p} + \mathbf{F}\mathbf{q}^{S-\{h\}}) = \sum_{j \in S-\{h\}} \lambda_j^{S-\{h\}} \mathbf{a}_j; \mathbf{a}_j^T\mathbf{q}^{S-\{h\}} = b_j; j \in S - h\right\} \tag{6}$$

and the critical inequality:

$$\mathbf{a}_h^T\mathbf{q}^{S-\{h\}} \geq b_h, \quad \text{for every } h \in S. \tag{7}$$

From eqns (5)–(7) we conclude by subtracting and premultiplying by $(\mathbf{q}^S - \mathbf{q}^{S-\{h\}})^T$, that

$$(\mathbf{q}^S - \mathbf{q}^{S-\{h\}})^T\mathbf{F}(\mathbf{q}^S - \mathbf{q}^{S-\{h\}}) = \lambda_h^S(\mathbf{a}_h^T\mathbf{q}^{S-\{h\}} - b_h). \tag{8}$$

From inequality (7) and the positive-semidefiniteness of matrix **F** it follows that

$$\lambda_h^S \geq 0 \quad \text{for every } h \in S. \tag{9}$$

Thus for $\mathbf{q} = \mathbf{q}^S$ the negative gradient of the function to be minimized can be expressed as a non-negative linear combination of the outer normals to the boundary of the admissible domain. Then (see, e.g. [17]) $\mathbf{q}^S$ represents the solution of problem (1). Conversely, if $\mathbf{q}^S = \hat{\mathbf{q}}$, then $\lambda_j^S = \hat{\lambda} \geq 0$, for every $j \in S$ and thus from (5), (6) and (9), using (8), we conclude, that (7) is valid. This necessary and sufficient condition for optimality is to be imposed on every $\mathbf{q}^S$ with $V(\mathbf{q}^S) = \emptyset$, when **F** is positive-semidefinite.

Finally, some additional comments are offered on the existence of the solution in the case of a positive-semidefinite matrix **F**. These are special cases of more general considerations which have been proved by Fremond ([8], pp. 123–128) for the continuous minimization problem.

Suppose, for instance, that:

(a) $b_j \geq 0 \; \forall_j \in S_0$

(b) either the set $A = \{\mathbf{q} | \mathbf{a}_j^T\mathbf{q} \leq b_j, \forall j \in S_0, \mathbf{F}\mathbf{q} = \mathbf{0}\}$ is bounded, or $\forall \mathbf{q} \in A$, such that for $\mathbf{q} \neq \mathbf{0}$ and $0 \leq \mu < \infty$ $\mu\mathbf{q} \in A$, $\mathbf{p}^T\mathbf{q} < 0$. Then problem (1) allows for at least one solution $\hat{\mathbf{q}}$. Moreover, the set $\hat{S}$ of the constraints, which was satisfied in equality form, is non-empty, if there exists a vector $\mathbf{q} \in A$ for which $\mathbf{p}^T\mathbf{q} \neq 0$. If there exist a vector $\mathbf{q} \in A, \mathbf{q} \neq \mathbf{0}$, such that for $0 \leq \mu \leq \infty$, $\mu\mathbf{q} \in A$ (equivalently $\mathbf{a}_j^T\mathbf{q} < 0 \; \forall_j \in S_0$) and $\mathbf{p}^T\mathbf{q} > 0$, then problem (1) does not have a solution.

The condition $\mathbf{p}^T\mathbf{q} < 0$ is called strong Signorini condition. It should be pointed out that if the weak Signorini condition $\mathbf{p}^T\mathbf{q} \leq 0$ instead of the strong one is valid the existence of the solution is not guaranteed by the previous assumptions. Denoting

$$K = \{\mathbf{q} | \mathbf{a}_j^T \mathbf{q} \le b_j \ \forall j \in S_0\}, \quad P = \{\mathbf{q} | \mathbf{q}^T \mathbf{p} = 0, \mathbf{F}\mathbf{q} \times \mathbf{0}\}$$

and

$$A' = (K + P) \cap \{\mathbf{q} | \mathbf{F}\mathbf{q} = \mathbf{0}\}$$

and assuming

(a) $b_j \ge 0 \ \forall j \in S_0$ ($k + P$ is a closed set)

(b) $\forall \mathbf{q} \in A'$, such that for $\mathbf{q} \ne \mathbf{0}$ and $0 \le \mu \le \infty \ \mu\mathbf{q} \in A', \mathbf{p}^T\mathbf{q} \le 0$ one finds problem (1) to have a solution.

3. RANGE OF APPLICABILITY

The method proposed in the present paper can be applied to solve a variety of unilateral problems which otherwise require quadratic optimization. The problems involve the unilateral stresses in cable structures, the unilateral contact of elastic bodies with or without friction, the holonomic and the incremental elastoplastic behavior of structures[15], etc. To apply the method developed here it is necessary to identify the U.C.'s, B.C.'s and the equality constraints imposed by the physical circumstances. Some specific problems serve here to illustrate the method.

3.1 *Stress-unilateral analysis of cable structures*

We consider structures containing some members which are capable of transmitting only tension (like cable-elements for example). The vector of stress s can be decomposed into the vector $\bar{\mathbf{s}}$, which involves the stresses of the elements with unilateral behavior and also the vector $\bar{\bar{\mathbf{s}}}$, the stress vector of the remaining elements with bilateral behavior. The vector $\bar{\mathbf{s}}$ satisfied then the unilateral constraint

$$\bar{\mathbf{s}} \geqq \mathbf{0}. \tag{10}$$

For the sake of brevity, physical and geometrical linearity is assumed. The generalization for cases of physical and geometrical nonlinearity is straightforward, when the updated Lagrangean formulation is used. It is shown in[18] that the problem can be formulated in terms of stresses as the following minimization problem:

$$\min\left[\Pi(\mathbf{s}) = \frac{1}{2}\mathbf{s}^T\mathbf{F}_0\mathbf{s} + \mathbf{s}^T\mathbf{e}_0 \middle| \mathbf{G}\mathbf{s} = \mathbf{p}, \bar{\mathbf{s}} \geqq \mathbf{0}\right]. \tag{11}$$

Let $\mathbf{s}_0$ denote a particular solution of the equilibrium equations:

$$\mathbf{G}\mathbf{s}_0 = \mathbf{p} \quad (\mathbf{p}\text{: load vector}). \tag{12}$$

The vector s can be written in the form

$$\mathbf{s} = \mathbf{s}_0 + \mathbf{B}\mathbf{x}, \text{ with } \mathbf{G}\mathbf{B} = \mathbf{0}.$$

The vector $\mathbf{s}_0$ can also be split into $\bar{\mathbf{s}}_0$ and $\bar{\bar{\mathbf{s}}}_0$ and also the matrix $\mathbf{B}$ into $\bar{\mathbf{B}}$ and $\bar{\bar{\mathbf{B}}}$. Thus the formulation of the problem in terms of forces can be stated in the following form:

$$\min\left[Q(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{D}\mathbf{x} + \mathbf{x}^T(\mathbf{d}_0 + \mathbf{B}^T\mathbf{e}_0) \middle| \bar{\mathbf{B}}\mathbf{x} + \bar{\mathbf{s}}_0 \geqq \mathbf{0}\right], \tag{13}$$

where

$$\mathbf{D} = \mathbf{B}^T\mathbf{F}_0\mathbf{B} \quad \text{and} \quad \mathbf{d}_0 = \mathbf{B}^T\mathbf{F}_0\mathbf{s}_0.$$

The quadratic optimization problem (13) can be solved as follows: The transformation of all of the U.C.'s $\bar{B}x + \bar{s}_0 \geq 0$ to B.C.'s consists of replacing the cables by pin-ended bars with the same flexibility. The resulting structure is called "respective bilateral structure". A constraint satisfied in equality form, i.e. $s_i = 0$ is realized by omitting the corresponding member. By means of the physical interpretation, the three rules give all the modifications (cutting of elements), which are successively performed on the "respective bilateral structure" and lead to the solution of the initial unilateral structure.

3.2 *Unilateral contact problems*

According to the type of contact and the material properties of the contiguous bodies, a variety of unilateral problems can be formulated. We restrict ourselves to "dry contact" problems, exclude lubricated surfaces and physical nonlinearities. The various kinds can be arranged in the following groups.

(a) *Boundary conditions of Signorini–Fichera type.* If a part of the body I rests on a rigid foundation, then the following unilateral condition holds (see Figs. 1a, c):

$$\text{if} \quad u_N < 0 \quad \text{then} \quad S_N = 0 \tag{14a}$$

$$\text{if} \quad u_N = 0 \quad \text{then} \quad S_N \leq 0. \tag{14b}$$

The terms Γ_u, Γ_s, Γ_f of Fig. 1(a) denote nonoverlapping parts of the boundary, where displacements, forces and unilateral conditions are prescribed respectively. Moreover we assume that on Γ_s the vector S_T or u_T are prescribed. On the constraints (14a) and (14b) a B.C. implies the removal of the supporting surface and an equality constraint implies contact. When a foundation lies at a variable distance $d(s)$ from the body, i.e. $u_N \leq d(s)$, an equality constraint is equivalent to the condition $u_N = d(s)$.

(b) *Unilateral constraints by elastic supports.* The following conditions describe the problem:

$$\text{if} \quad u_N < 0 \quad \text{then} \quad S_N = 0 \tag{15a}$$

$$\text{if} \quad u_N \geq 0 \quad \text{then} \quad S_N = -ku_N, \tag{15b}$$

where k denotes the Winkler's constant of the tensionless foundation. These conditions are completed by the condition $S_T = C_T$ on Γ_S. The first condition (15a) holds at the regions, where the support is not attached, and hence it cannot exert tension, and the second condition holds at the

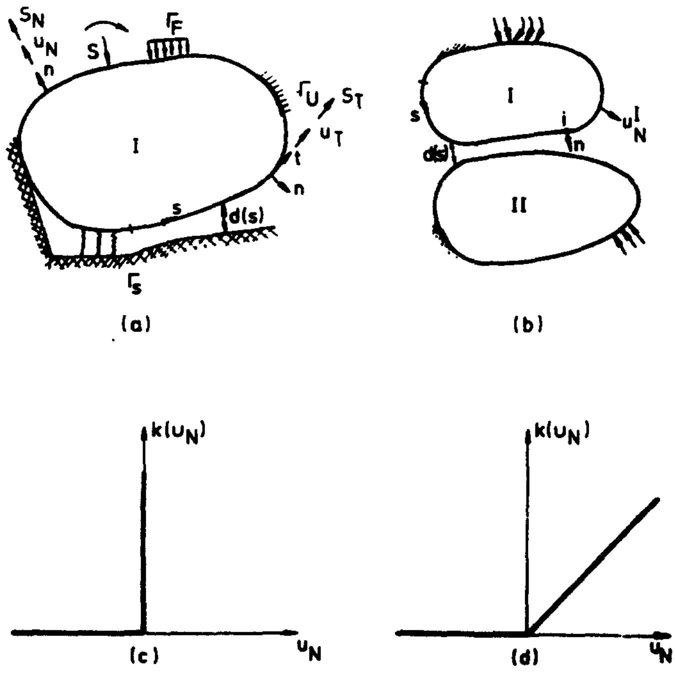

Fig. 1. Problems with unilateral constraints.

contact regions (see Figs. 1a and 1d). A B.C. is realized by attaching the spring to the body and an equality constraint by omitting the spring.

(c) *Friction boundary conditions.* Contact between dry surfaces may be described by the Coulomb's law: If f denotes the Coulomb's coefficient of friction and λ a positive unspecified coefficient of proportionality, then the friction boundary conditions take the form:

$$\text{if} \quad |S_T| < f|S_N|, \quad \text{then} \quad u_{Ti} = 0 \tag{16a}$$

$$\text{if} \quad |S_T| = f|S_N|, \quad \text{then} \quad u_{Ti} = -\lambda S_{Ti}, \lambda \geqq 0. \tag{16b}$$

The conditions (16) introduce nondifferentiable terms in the functional of the potential energy to be minimized. The solution requires the use of special optimization algorithms, which avoid the calculation of derivatives[13]. However, the present method can be applied as well to the solution of the friction problems, if one considers the complementary energy functional. In this case a quadratic optimization problem of the form (1) arises[13]. If the constraints (16) occur together with the constraints (14), the following method of approximation can be applied: As a starting point, the problem without friction is solved and the forces S_{N1} are estimated. For $S_N = S_{N1}$ the friction problem is solved and the value of S_T, say S_{T1}, is calculated. Solving the unilateral problem for $C_T = S_{T1}$ a new set of forces S_{N2} is obtained. The procedure is repeated until the differences $S_{T_{i+1}} - S_{T_i}$ and $S_{N_{i+1}} - S_{N_i}$, respectively, fall within prescribed limits. By means of this procedure the sliding and adhesive friction regions are determined together with the contact and non-contact regions.

3.3 *Two elastic bodies in contact*

At any point i (Fig. 1b) in the zone of contact, the difference between the displacements of the adjacent bodies must be less than the initial separation $d(s)$. This condition is stated as follows:

$$u_N^I - u_N^{II} \leqq d(s). \tag{17}$$

Such conditions of compatibility are included in the general form:

$$q_k = \mathbf{A}_k \mathbf{q}_i + \delta_k, \tag{18}$$

where δ_k is a given value and A is a prescribed matrix relating the variables q_k and $\mathbf{q}_i$.

4. SPECIAL CONSIDERATIONS

To perform an iteration step, any finite element program may be utilized. Superfluous computational effort can be avoided if the existing programs are slightly modified in order to treat the constraints satisfied in equality form in an effective manner. The equality constraints lead to relations of the following general form:

$$q_r = \bar{q}_r + \sum_{\substack{p=1 \\ p \neq r}}^{n} C_p q_p, \tag{19}$$

where q_r denotes the rth degree of freedom which is coupled with m variables q_p, and the coefficients C_p are elements of an $1 \times (n-1)$ vector containing m nonzero elements. In the case of the Signorini–Fichera conditions eqn (19) takes the simpler form $q_r = \bar{q}_r$. The introduction of the equality constraints into the unconstrained problem leads to a nonsymmetrical matrix. This can be avoided by performing some matrix operations proposed in[19]. In other respect the present procedure differs: The dependent variables are not condensed out but retained, the original matrix is stored and each equality constraint is introduced successively. This avoids a reordering of the unknowns and is suitable when using equation solvers taking into account the blockstructure of the system matrix.

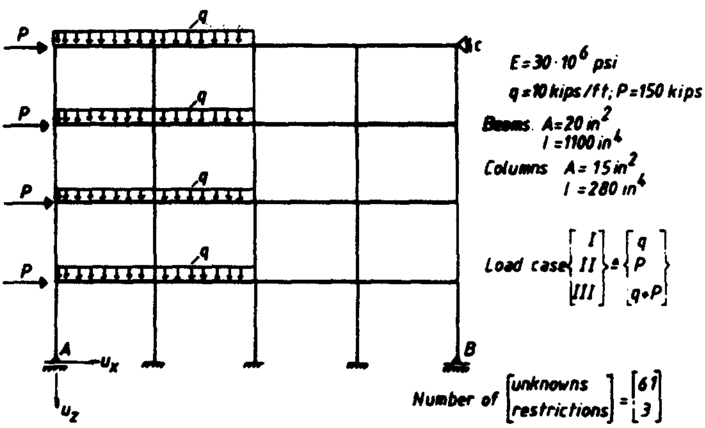

Fig. 2. Frame with unilateral supports.

5. EXAMPLE PROBLEMS

The following examples provide insight into the proposed method and also demonstrate its applicability and convergence.

(a) *First example*

The frame shown in Fig. 2 involves three unilateral supports with the constraints $u_{z_i} \leq 0$, $i = A, B$ and $u_{x_C} \leq 0$. The problem is solved in[10] with the aid of an optimization algorithm. Here details of the present procedure are described for the load case III.

As a first step, the U.C.'s A, B, C are replaced by B.C.'s, i.e. the problem is solved without the above mentioned supports. The solution $\dot{\mathbf{q}}$ violates all restrictions, i.e. $V(\dot{\mathbf{q}}) \neq \emptyset$. According to Rule 1, the correct solution $\hat{\mathbf{q}}$ must satisfy, in equality form, at least one of the violated restrictions. Therefore, the restriction imposed by support A is replaced by an equality constraint, i.e. at A the constraint $u_{z_A} = 0$ is introduced. The solution $\mathbf{q}^A$ of this system yields a set of violations: $V(\mathbf{q}^A) = \{B, C\}$. Following Rule 2, a constraint is introduced at C. This leads to $V(\mathbf{q}^S) = \emptyset$, where $S = \{A, C\}$. The next step is the test for optimality of solution $\mathbf{q}^S$: Each of the equality constraints introduced previously must now be successively removed and each corresponding solution must violate that constraint ($h \in V(\mathbf{q}^{S-\{h\}})$ for every $h \in S$). The solution $\mathbf{q}^{S-\{A,C\}}$ was obtained by satisfying the restrictions at A, C in equality form. Therefore, the equality constraint in A is first removed to obtain a solution $\mathbf{q}^{S-\{h\}} = \mathbf{q}^{\{A,C\}-A} = \mathbf{q}^C$ which violates the condition at A. Finally, the equality constraint in C is removed to obtain the solution $\mathbf{q}^{S-\{h\}} = \mathbf{q}^{\{A,C\}-C} = \mathbf{q}^A$ with a set $V(\mathbf{q}^A)$ containing C. According to Rule 3, the optimal solution for load case III is $\mathbf{q}^{A,C}$ and thus the structure has contact at A, C. Table 1 shows the results obtained for the three load cases which agree with those given in[10]. It is apparently difficult to guess the role of the unilateral support under different load conditions: e.g. under the forces P acting on the frame, it is not obvious that support A maintains contact and not support B.

(b) *Second example*

The next example (Fig. 3) demonstrates the applicability of the method to problems with finite areas of contact and also the importance of the test given in Rule 3. Two conditions of

Table 1. Frame with unilateral supports—results for load cases I–III

Load case	Contact with support	u_z^A	u_z^B	u_x^C	Reaction [Kips] at A	B	C
I	A,B,C	0.0	0.0	0.0	365.720	0.885	0.892
II	A,C	0.0	-1.13719	0.0	45.485	0.0	379.058
III	A,C	0.0	-1.11586	0.0	411.168	0.0	380.158

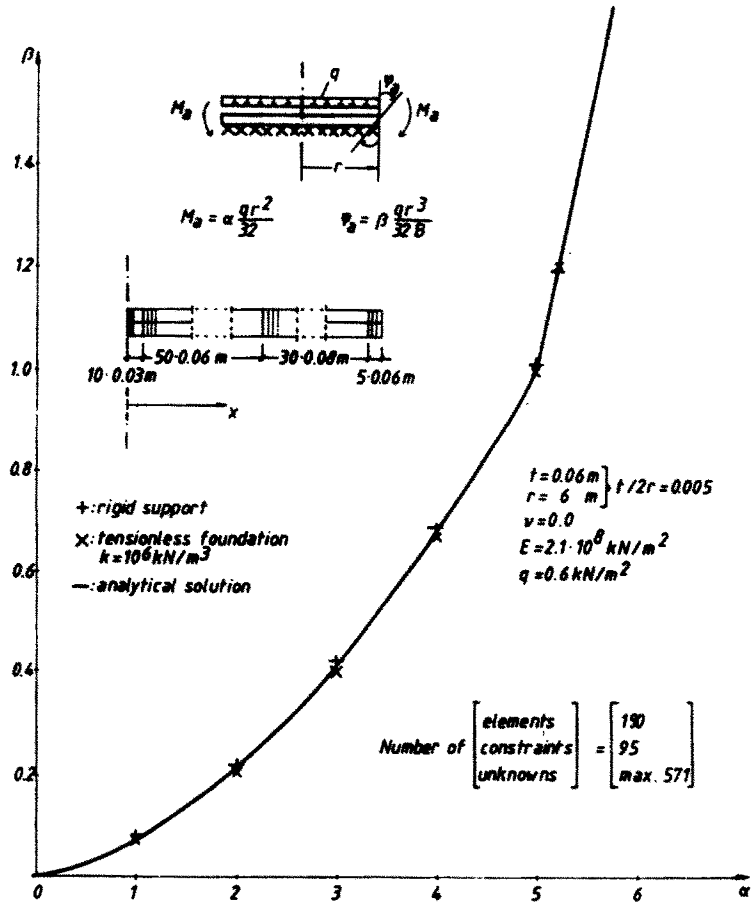

Fig. 3. Circular plate—analytical and numerical solutions.

support are treated: Firstly, the plate rests upon a rigid plane and secondly, upon a tensionless foundation. The conditions are comparable to the actual conditions at the ends of cylindrical tanks. Analytical solutions for the plate on the rigid foundation are given in [20]. Here the case of tensionless foundation is also investigated.

The essential steps of the method are described for the case $\alpha = 1$. The first step–solution of a simply supported circular plate—violates all restrictions. In the next iteration equality constraints are simultaneously introduced at a number of points within a preselected region, i.e. points lying between $x = 0.0$ m and $x = 3.3$ m. Since, the solution gives a nonempty set of violated restrictions, the equality constraint are extended in the third step to $x = 3.62$ m and

Table 2. Circular plate resting on a rigid foundation

α	M_a	$\varphi_\alpha 10^{-5}$ *)	$\varphi_\alpha 10^{-5}$ *)	β	contact+*) area [m]	contact**) between [m]
1	0.675	7.29036	-7.5	0.06804336	3.70	3.70/3.78
2	1.35	22.2363	-23.0	0.2075391	2.58	2.58/2.64
3	2.025	43.8	-43.9	0.4088	1.56	1.56/1.62
4	2.7	72.37966	-71.5	0.67554	0.54	0.54/0.60
5	3.375	107.176	-107.14	1.0003	0.0	0.0
5.2	3.51	128.592	-129.0	1.200192	no contact	

+*) discrete solution
**) continuous solution
*) present paper
+) taken from given figure

Table 3. Circular plate resting on an elastic tensionless foundation

α	M_a	$\varphi_a \cdot 10^{-6}$	β	no contact* between $x=$	and $x=$	no contact** between $x=$	and $x=$
1	0.675	87.017667	0.08121649	4.10	5.82	4.02/4.10	5.82/5.88
2	1.35	240.328	0.224306	2.94	5.88	2.88/2.94	5.88/5.94
3	2.025	458.22667	0.427678	1.92	5.88	1.86/1.92	5.88/5.94
4	2.7	741.87667	0.692418	0.96	5.88	0.90/0.96	5.88/5.94
5	3.375	1084.59	1.012284	0.18	5.94	0.15/0.18	5.94/6.00
5.01	contact at $x = 0.0$ [m]						
5.2	3.51	1285.92	1.20019	no contact			
6.0	4.05	2143.10	2.000227				

*discrete solution
**continuous solution

again, for the same reason in the fourth step to $x = 3.78$ m. This system does not violate the restrictions but its solution—after application of the test—is not the correct solution. Introduction of equality constraints to $x = 3.7$ m leads to an empty set of violated restrictions. Rule 3 requires the application of the test for all equality constraints introduced previously. Here the test need be applied only for the exterior point at $x = 3.7$ m, because it is apparent, that all interior points will pass the test. Thus much computational effort can be avoided by the simultaneous use of several equality constraints and by applying the test only as dictated by the physical circumstances. Due to the continuous change of the contact area the Moment-Rotation diagram of Fig. 3 is nonlinear to $\alpha = 5$. The stiffness of the system decrease with decreasing contact zone. At $\alpha = 5$ the plate lifts entirely from the support ant the relationship becomes linear. The comparison of the results with the analytical solution (Table 2) show a good agreement.

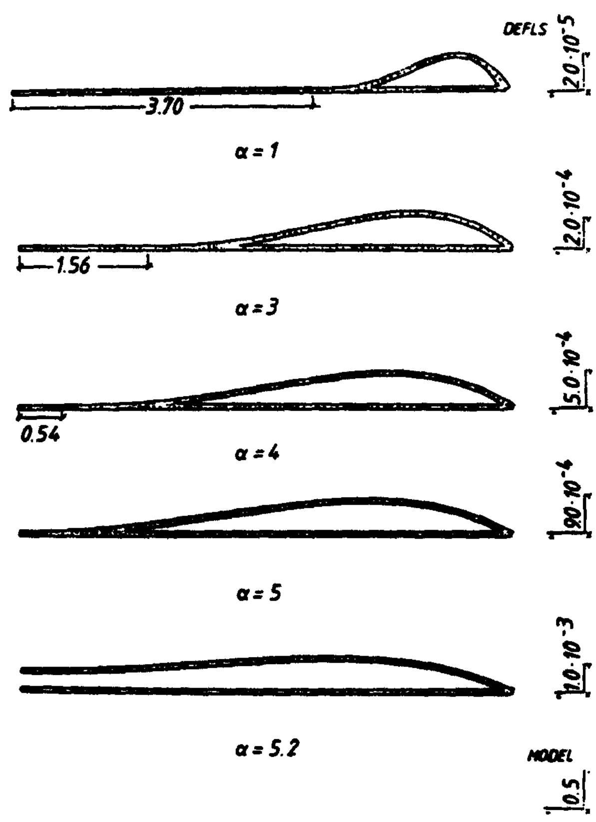

Fig. 4. Circular plate resting on a rigid foundation—deformed shapes.

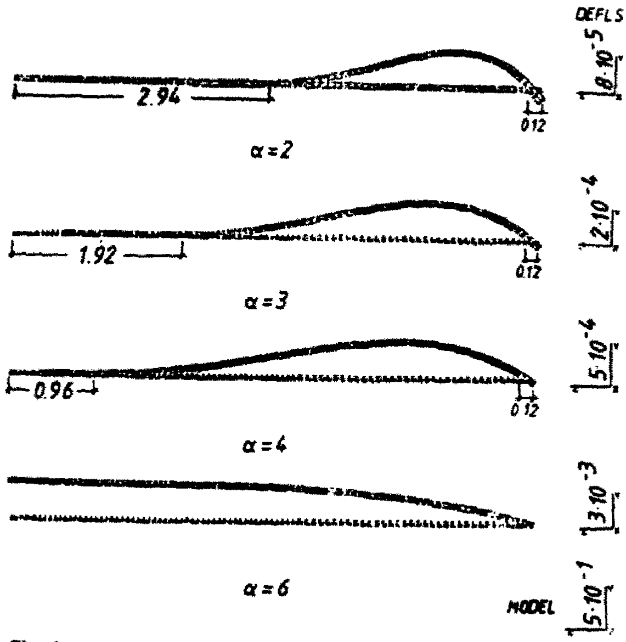

Fig. 5. Circular plate resting on an elastic tensionless foundation—deformed shapes.

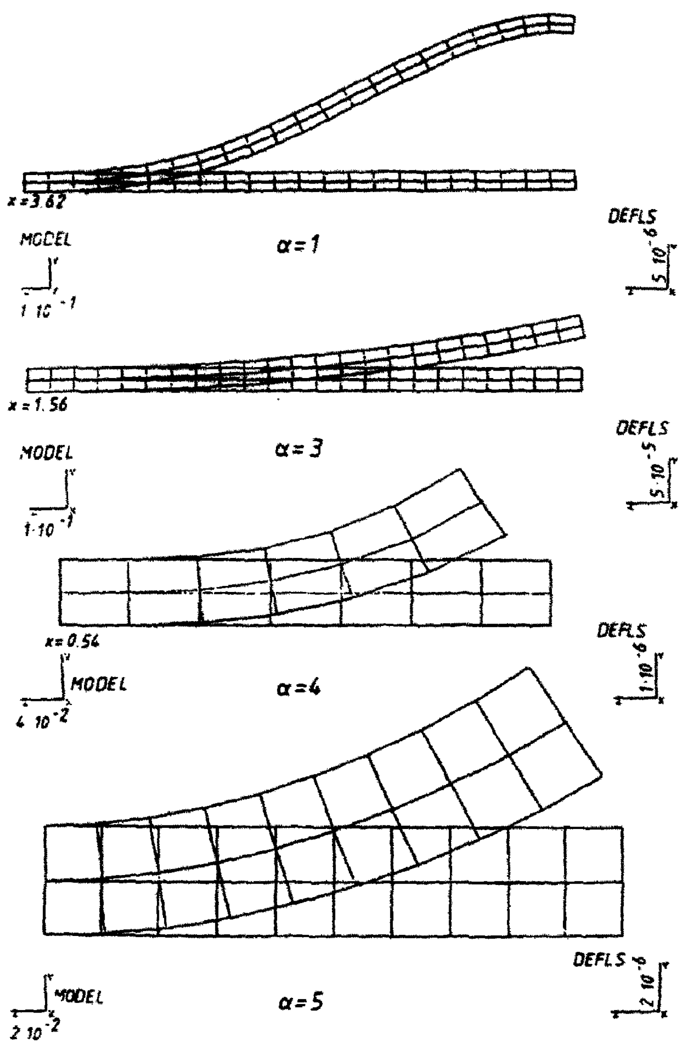

Fig. 6. Circular plate resting on a rigid foundation.

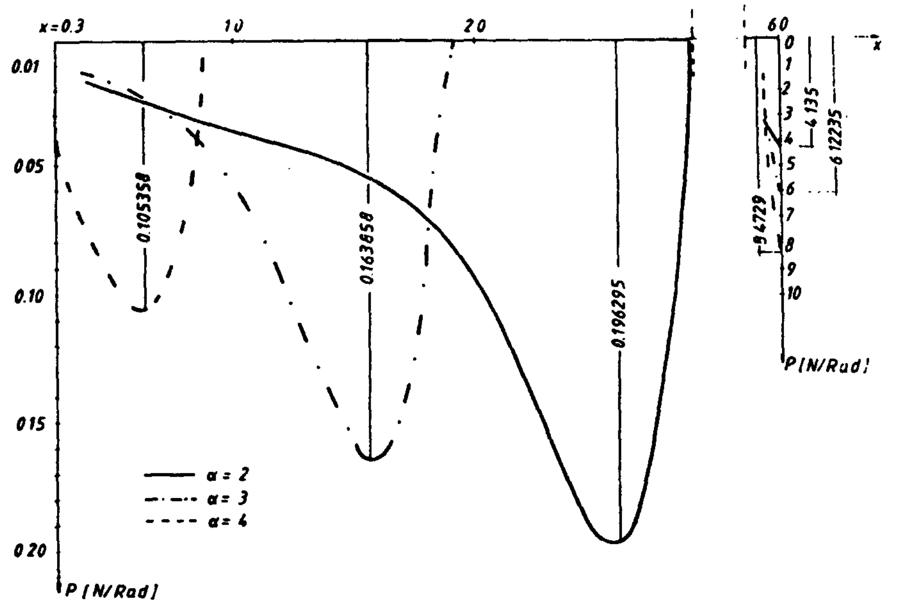

Fig. 7. Circular plate resting on an elastic tensionless foundation—contact forces.

Finally, the response of a plate on a unilateral elastic foundation is investigated. The foundation is idealized by discrete springs corresponding to Winkler's model. In contrast with the rigid support, the contact zone consists of two parts: Contact in a central zone and an edge zone. This leads to an increase in the required number of iterations and clearly shows the importance of performing the test. Tables 2 and 3 show the results obtained for both cases and Figs. 4–6 illustrate the deformation shapes of the plate. Finally, the distribution of the contact forces for several values of the bending moment M_a are illustrated in Fig. 7.

(c) *Third example*

The cylindrical tube of Fig. 8 is encased in concrete, assumed rigid and subjected to external hydrostatic pressure. The cylinder is approximated in two ways: Flat elements are employed with a displacement formulation and a curved shell element based on a mixed formulation[21] in terms of stresses and displacements. The latter approach demonstrates the applicability of the method in connection to mixed finite element models, i.e. the present method could be seen as an attempt towards the use of mixed finite elements for the numerical solution of unilateral problems arising in structural mechanics. This example is investigated in[22] which also considers geometrical nonlinearities and uses "trial and error" methods. Here geometrical nonlinearities and friction are neglected. Also the cylinder is assumed long enough so that end effects are not considered and the behavior is independent of the axial coordinate. The problem

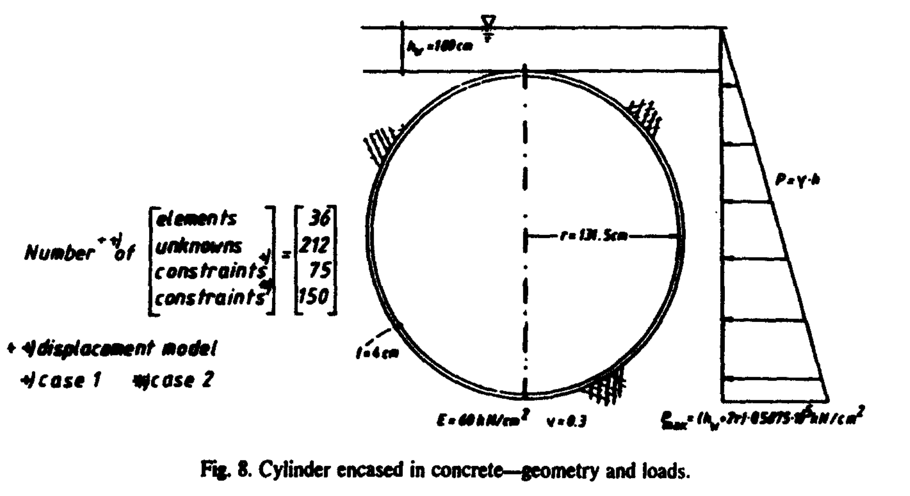

Fig. 8. Cylinder encased in concrete—geometry and loads.

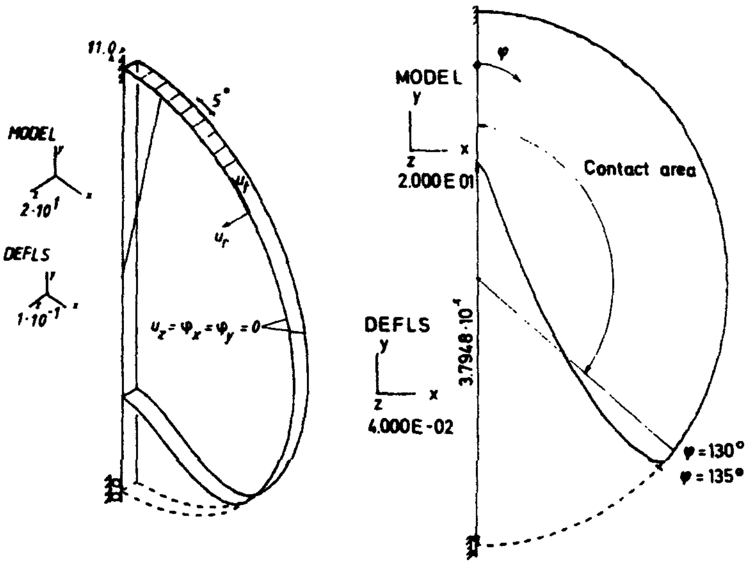

Fig. 9. Cylinder encased in concrete—second case.

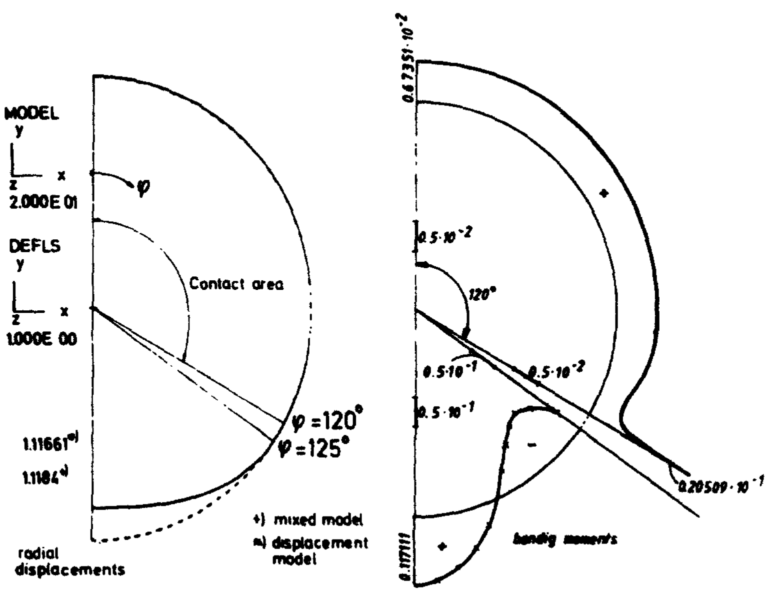

Fig. 10. Cylinder encased in concrete—first case.

is solved for two cases of constraints. In the first case only the radial displacements are constraint against outward movement and the tangential displacements are unrestricted. In the second case the tangential displacements are also prohibited when contact is maintained, i.e. radial displacements vanish. In our first case upward radial displacements are inadmissible, but the solution admits small tangential displacements. In our second case, tangential displacements as well as radial displacements are prohibited on the contacting surfaces; the constraints lead to an increase in the stiffness of the system. Figures 9 and 10 illustrate the deformed shape of the shell and the contact area. Figure 10 also shows the distribution of bending moments.

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